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Lower Bounds for the Product of the Eigenvalues of the Solutions to tihe Matrix Equations

Masoud Hajarian[a]

[a]*Department of Mathematics, Faculty of Mathematical Sciences, Shahid Beheshti University, General Campus, Evin, Tehran 19839, Iran.*

Abstract. In this paper, we propose the inequalities which are satisfied by the determinant of the solutions to the systems of matrix equations

$$\begin{cases} AXB^T + BX^TA^T = C, \\ DXE^T + EX^TD^T = F, \end{cases} \qquad \text{(I)}$$

$$\begin{cases} AXB^T + BXA^T = C, \\ DXE^T + EXD^T = F. \end{cases} \qquad \text{(II)}$$

By applying the proposed inequalities, lower bounds for the product of the eigenvalues of the solutions to the systems of matrix equations (I) and (II) are established.

1. Introduction

The matrix equations can be encountered in many areas of computational mathematics and engineering. The matrix equations such as the Lyapunov and Sylvester matrix equations have a close relation with many problems in linear control theory of descriptor systems [1]. Therefore a large number of papers have studied the matrix equations [2–5, 13]. So far, various bounds for the eigenvalues of solutions of several matrix equations have been introduced. For the Lyapunov matrix equation

$$A^TP + PA = -Q, \tag{1.1}$$

Shapiro in [11] presented a fundamental equalities relating the smallest eigenvalue of matrix Q to the smallest eigenvalue of matrix P. In [7], upper bounds for summations including the trace, and for products including determinant, of the eigenvalues of the solution the continuous algebraic Lyapunov matrix equation (1.1) were presented. Lower bounds of the solution K to the Riccati matrix equation

$$A^TK + KA - KBB^TK + Q = 0, \tag{1.2}$$

were proposed in [9]. Komaroff obtained lower bounds for the sum of the eigenvalues of the solution to the algebraic matrix Riccati equation [6]. In [8], several bounds for the trace of the solutions of the

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Email address: m_hajarian@sbu.ac.ir, mhajarian@aut.ac.ir, masoudhajarian@gmail.com (Masoud Hajarian)

algebraic Riccati and Lyapunov matrix equations were presented in the continuous and discrete domain, respectively. Also lower and upper bounds on the eigenvalues of the Lyapunov and Riccati matrix equations were obtained in [10].
In this paper, we consider the following two systems of matrix equations

$$\begin{cases} AXB^T + BX^TA^T = C, \\ DXE^T + EX^TD^T = F, \end{cases} \tag{1.3}$$

$$\begin{cases} AXB^T + BXA^T = C, \\ DXE^T + EXD^T = F, \end{cases} \tag{1.4}$$

where $A, B, C, D, E, F \in \mathcal{R}^{n\times n}$ are known matrices and $X \in \mathcal{R}^{n\times n}$ is unknown. The systems contain several linear matrix equations such as the Lyapunov and Sylvester matrix equations. We will introduce the inequalities which are satisfied by the determinant of the solutions of the systems (1.3) and (1.4), respectively. Then using the proposed inequalities, we introduce lower bounds for the product of eigenvalues of solutions to the systems.
In this paper, we will use the following notations. We denote the set of $n \times n$ matrices by $\mathcal{R}^{n\times n}$. The symbols A^T, $|A|$ and $\lambda(A)$ denote the transpose, the determinant and the spectrum of matrix A, respectively. Matrix $A \in \mathcal{R}^{n\times n}$ is said to be a stability matrix (or stable matrix) if $\mathrm{Re}(\lambda_i) < 0$ for $i = 1, 2, ..., n$, where $\lambda(A) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$. If A is a square n-by-n matrix with $\lambda(A) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$, then $|A| = \lambda_1\lambda_2...\lambda_n$.
In the next section, we will present main results.

2. Main Results

In the rest of this paper, we will suppose that C and F are symmetric positive definite matrices. In this section, to facilitate the statement of our main results, we give the following lemma that its proof can be found in [12].

Lemma 1. *Assume that $E = F + G$ and F is a symmetric positive definite matrix and G is a skew-symmetric matrix, then $|E| \geq |F|$.*

Theorem 1. *Suppose that the system of matrix equations* (1.3) *is consistent, then*

$$|C| \leq 2^n|X||A||B|, \qquad \text{and} \qquad |F| \leq 2^n|X||D||E|. \tag{2.1}$$

Proof. Considering the system of matrix equations (1.3) gives us

$$\begin{aligned} AXB^T &= \frac{1}{2}[AXB^T + AXB^T + BX^TA^T - BX^TA^T] \\ &= \frac{1}{2}(AXB^T + BX^TA^T) + \frac{1}{2}(AXB^T - BX^TA^T), \end{aligned}$$

and

$$\begin{aligned} DXE^T &= \frac{1}{2}[DXE^T + DXE^T + EX^TD^T - EX^TD^T] \\ &= \frac{1}{2}(DXE^T + EX^TD^T) + \frac{1}{2}(DXE^T - EX^TD^T). \end{aligned}$$

By defining the matrices

$$G = \frac{1}{2}(AXB^T - BX^TA^T) \qquad \text{and} \qquad H = \frac{1}{2}(DXE^T - EX^TD^T),$$

and using (1.3) we can write

$$AXB^T = \frac{1}{2}C + G \qquad \text{and} \qquad DXE^T = \frac{1}{2}F + H. \tag{2.2}$$

Obviously, the matrices G and H are skew-symmetric matrices. By applying Lemma 1, we have

$$\begin{aligned} &|AXB^T| \geq |\frac{1}{2}C|, \\ &|A||X||B| \geq \frac{1}{2^n}|C|, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &|DXE^T| \geq |\frac{1}{2}F|, \\ &|D||X||E| \geq \frac{1}{2^n}|F|. \end{aligned} \tag{2.4}$$

Now from (2.3) and (2.4), the conclusion (2.1) holds. □

By a similar proof to the previous theorem we can prove the following theorem.

Theorem 2. *Suppose that the system of matrix equations* (1.4) *is consistent over the symmetric matrix* $X \in \mathcal{R}^{n\times n}$*, then*

$$|C| \leq 2^n|X||A||B|, \qquad \text{and} \qquad |F| \leq 2^n|X||D||E|. \tag{2.5}$$

Theorem 3. *Assume that the system of matrix equations* (1.3) *is consistent and* $A, B, D, E \in \mathcal{R}^{n\times n}$ *are stability matrices. If* $\lambda(A) = \{\alpha_1, \alpha_2, ..., \alpha_n\}$*,* $\lambda(B) = \{\beta_1, \beta_2, ..., \beta_n\}$*,* $\lambda(C) = \{\gamma_1, \gamma_2, ..., \gamma_n\}$*,* $\lambda(D) = \{\zeta_1, \zeta_2, ..., \zeta_n\}$*,* $\lambda(E) = \{\eta_1, \eta_2, ..., \eta_n\}$*,* $\lambda(F) = \{\xi_1, \xi_2, ..., \xi_n\}$ *and* $\lambda(X) = \{\delta_1, \delta_2, ..., \delta_n\}$*, then*

$$\frac{1}{2^n}\max\Big\{\frac{\prod_{i=1}^n \gamma_i}{\prod_{i=1}^n \alpha_i\beta_i}, \frac{\prod_{i=1}^n \xi_i}{\prod_{i=1}^n \zeta_i\eta_i}\Big\} \leq \delta_1\delta_2...\delta_n. \tag{2.6}$$

Proof. From A, B, D and E are stability matrices, we can get

$$|A| > 0,\ \ |B| > 0,\ \ |D| > 0,\ \ |E| > 0,\ \ \text{for } n \text{ even},$$

and

$$|A| < 0,\ \ |B| < 0,\ \ |D| < 0,\ \ |E| < 0,\ \ \text{for } n \text{ odd}.$$

Then $|A||B| > 0$ and $|D||E| > 0$ for $n = 1, 2, 3, ...$ From $|A||B| > 0$ and $|D||E| > 0$, we can write

$$\frac{|C|}{2^n|A||B|} \leq |X|, \qquad \text{and} \qquad \frac{|F|}{2^n|D||E|} \leq |X|. \tag{2.7}$$

Now by (2.7), we have

$$\frac{1}{2^n}\max\Big\{\frac{\prod_{i=1}^n \gamma_i}{\prod_{i=1}^n \alpha_i\beta_i}, \frac{\prod_{i=1}^n \xi_i}{\prod_{i=1}^n \zeta_i\eta_i}\Big\} \leq \delta_1\delta_2...\delta_n.$$

The proof is completed. □

Similarly to the proof of Theorem 4, we can prove the following theorem.

Theorem 4. *Assume that the system of matrix equations* (1.4) *is consistent over the symmetric matrix* $X \in \mathcal{R}^{n\times n}$ *and* $A, B, D, E \in \mathcal{R}^{n\times n}$ *are stability matrices. If* $\lambda(A) = \{\alpha_1, \alpha_2, ..., \alpha_n\}$*,* $\lambda(B) = \{\beta_1, \beta_2, ..., \beta_n\}$*,* $\lambda(C) = \{\gamma_1, \gamma_2, ..., \gamma_n\}$*,* $\lambda(D) = \{\zeta_1, \zeta_2, ..., \zeta_n\}$*,* $\lambda(E) = \{\eta_1, \eta_2, ..., \eta_n\}$*,* $\lambda(F) = \{\xi_1, \xi_2, ..., \xi_n\}$ *and* $\lambda(X) = \{\delta_1, \delta_2, ..., \delta_n\}$*, then*

$$\frac{1}{2^n}\max\Big\{\frac{\prod_{i=1}^n \gamma_i}{\prod_{i=1}^n \alpha_i\beta_i}, \frac{\prod_{i=1}^n \xi_i}{\prod_{i=1}^n \zeta_i\eta_i}\Big\} \leq \delta_1\delta_2...\delta_n. \tag{2.8}$$

3. Concluding Remarks

In this paper, we have presented the inequalities for the determinant and the product of eigenvalues of solutions to systems (1.3) and (1.4). We have showed by knowing $|A|$, $|B|$, $|C|$, $|D|$, $|E|$ and $|F|$, the lower bounds for the determinant and the product of eigenvalues of solutions to systems can be obtained.

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